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Application of Complex Numbers to Geometry

The paper includes needed pieces of information and results, the adoption of which shows a wide range of opportunities for doing research in the area of application of complex numbers. It consists of two parts.

The basic terms and definitions related to complex numbers are defined at the beginning of each section. The ways of interpretation of various geometric facts are brought out. Most of the sections contain the solutions to problems on the covered topic.

Problems given at Olypmiads and competitions, that are solved easily by means of complex numbers, are included as examples of the theory. Some problems are only stated but then follows the demonstration of a more general statement.

1. Introduction and basic definitions

Complex numbers can be interpreted as points in a rectangular coordinate system. The position of each point Z in the plane is fixed by its two coordinates (a, b) (in the rectangular coordinate system). Each point $Z(a, b)$ from the plane can be juxtaposed with a complex number $z = a + bi$ (i – the imaginary unit, $i^2 = -1$; a, b – real numbers). Now follow the definitions of equality of two complex numbers and the operations addition, subtraction and multiplication of two complex numbers so that the basic laws of algebra are observed.

All theoretical results (definitions, concepts and statements) in this and in the following sections of the paper are taken from [2].

Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ be two complex numbers.

Definition 1. We say that z_1 and z_2 are equal if $a_1 = a_2$ and $b_1 = b_2$.

Definition 2. Arithmetic operations with complex numbers:

Addition: $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$;

Subtraction: $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$;

Multiplication: $z_1 z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$.

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Definition 3. If $z = a + bi$, the complex number $\overline{z} = a - bi$ is called conjugated of z.

Let $z = a + bi$ and $u = p + qi$ be complex numbers. The following equalities hold true:

$$\overline{z+u} = \overline{z} + \overline{u}, \quad \overline{z-u} = \overline{z} - \overline{u}, \quad \overline{zu} = \overline{z}\,\overline{u}, \quad \overline{\left(\frac{z}{u}\right)} = \frac{\overline{z}}{\overline{u}}$$

Each point in the plane corresponds to a complex number, which is called the affix of this point.

In order to deal with a pile of mathematical problems we have to define modulus and argument of a complex number $z = a + bi$.

Figure 1.

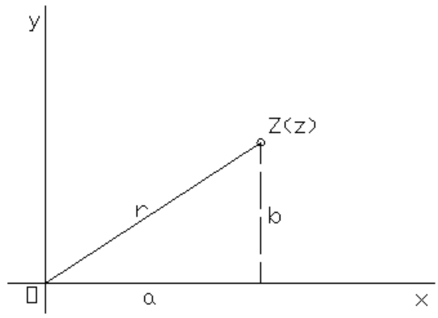
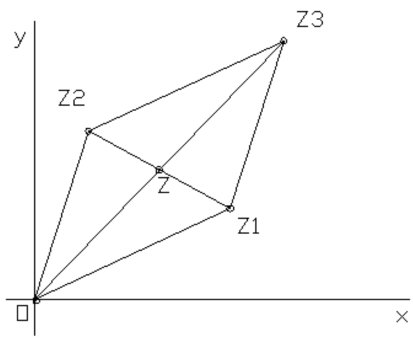


Figure 2.



The axis passing through the points Z_0 and Z_1 is specified with a directing vector $\overrightarrow{Z_0Z_1}$ and is denoted by Z_0Z_1. Oriented angle between the axes l_1 and l_2 is the angle describing the clockwise movement of the line l_1 to the position l_2 so that their positive directions coincide. This angle is defined with accuracy to a multiple of 2π.

Denote by r the length of the vector $\overrightarrow{OZ}$ (see Figure 1), and by φ the oriented angle between the positive direction of the horizontal coordinate axis Ox and the vector $\overrightarrow{OZ}$. The number r is real and non-negative. If we look at points on Ox, the affix of which is a real number, the number r is equal to the absolute value of the corresponding real. That is why r is called a modulus when talking about complex numbers and is denoted by $r = |z|$ ($r = \sqrt{a^2 + b^2}$ or $r^2 = a^2 + b^2 = z\,\overline{z}$).

The angle φ is called argument of z. The argument φ of the complex number $z = a + b\,i$ (denoted by arg z) can be obtained through the equalities (see Figure 1):

$$\cos\varphi = \frac{a}{r} = \frac{a}{\sqrt{a^2+b^2}} \qquad \text{and} \quad \sin\varphi = \frac{b}{r} = \frac{b}{\sqrt{a^2+b^2}}$$

The argument of number $z \neq 0$ has infinite number of values. If φ is one of these values, then all values can be obtained through the equality:

$$\arg z = \varphi + 2k\pi$$

where k can be any integer. We usually use one value of the argument, that in the interval $[0, 2\pi]$. It is denoted by Arg z and is called the main argument of z. Only the number 0 does not have an argument because it is defined only by its modulus $r = 0$.

2. Geometric presentation of complex numbers

In what follows, we shall denote the points in the plane by capital letters and their affixes by the corresponding lower case.

Complex numbers can be interpreted as vectors as well, that is the point $Z(z)$ coincides with a corresponding vector $\overrightarrow{OZ}$. Let $Z_1(z_1)$ and $Z_2(z_2)$ be two points in the plane. Then their sum $z = z_1 + z_2$ is a point Z, for which $\overrightarrow{OZ} = \overrightarrow{OZ_1} + \overrightarrow{OZ_2}$, and their difference $z_2 - z_1$ – the vector $\overrightarrow{OZ} = \overrightarrow{OZ_1} - \overrightarrow{OZ_2}$ (see Figure 2). The distance d between Z_1 and Z_2 or the length of the vector $\overrightarrow{Z_1Z_2}$ equals $d = |Z_1 Z_2| = |z_2 - z_1|$, i. e. d is the modulus of the complex number $z_2 - z_1$.

Let Z be the midpoint of the line segment Z_1Z_2. According to the rule of addition of vectors $\overrightarrow{OZ} = \dfrac{\overrightarrow{OZ_1} + \overrightarrow{OZ_2}}{2}$, or $z = \dfrac{z_1 + z_2}{2}$ are fulfilled. The oriented angle δ between the rays Oz_1 and Oz_2 is computed by choosing points Z_1 and Z_2 from each ray respectively, not coinciding with O and setting $\delta = \arg z_2 - \arg z_1 = \operatorname{Arg}\dfrac{z_2}{z_1}$. In the more general case, when the two rays are of the form Z_0Z_1 and Z_0Z_2 ($Z_0(z_0) \neq$ O), the oriented angle is:

$$\delta = \operatorname{Arg}(z_2 - z_0) - \operatorname{Arg}(z_1 - z_0) = \operatorname{Arg}\frac{z_2 - z_0}{z_1 - z_0}$$

Analogously,

$$\angle(Z_1 Z_2, U_1 U_2) =. Arg\frac{z_2 - z_1}{u_2 - u_1}$$

Now we will see when $\overrightarrow{Z_1Z_2} \perp \overrightarrow{U_1U_2}$. It is sufficient that $\frac{z_2 - z_1}{u_2 - u_1}$ is a purely imaginary number, i. e. the numbers $\frac{z_2 - z_1}{u_2 - u_1}$ and $\frac{\overline{z}_2 - \overline{z}_1}{\overline{u}_2 - \overline{u}_1}$ are equal:

$$\frac{z_2 - z_1}{u_2 - u_1} = \frac{\overline{z}_2 - \overline{z}_1}{\overline{u}_2 - \overline{u}_1}$$

$$(z_2 - z_1) \cdot (\overline{u}_2 - \overline{u}_1) = -(\overline{z}_2 - \overline{z}_1) \cdot (u_2 - u_1)$$

$$(z_2 - z_1) \cdot (\overline{u}_2 - \overline{u}_1) + (\overline{z}_2 - \overline{z}_1) \cdot (u_2 - u_1) = 0 \qquad (1)$$

The vectors $\overrightarrow{Z_1Z_2}$ and $\overrightarrow{U_1U_2}$ are parallel when Arg $\frac{z_2 - z_1}{u_2 - u_1}$ is 0° or 180°, which means that the number $\frac{z_2 - z_1}{u_2 - u_1}$ lies on the real axis *Ox*, or $\frac{z_2 - z_1}{u_2 - u_1}$ is a real number, that is:

$$(z_2 - z_1) \cdot (\overline{u}_2 - \overline{u}_1) = (\overline{z}_2 - \overline{z}_1) \cdot (u_2 - u_1) \qquad (2)$$

To summarize (1) and (2) we have:

$$\overrightarrow{Z_1Z_2} \perp \overrightarrow{U_1U_2} \Leftrightarrow (z_2 - z_1) \cdot (\overline{u}_2 - \overline{u}_1) + (\overline{z}_2 - \overline{z}_1) \cdot (u_2 - u_1) = 0 \quad (3)$$

$$\overrightarrow{Z_1Z_2} \parallel \overrightarrow{U_1U_2} \Leftrightarrow (z_2 - z_1) \cdot (\overline{u}_2 - \overline{u}_1) = (\overline{z}_2 - \overline{z}_1) \cdot (u_2 - u_1) \qquad (4)$$

Problem 1 *(XXIV Russian Olympiad)*. A hexagon *ABCDEF* is given with $\overrightarrow{BC} = \overrightarrow{EF}$. Consider the positively oriented isosceles right-angled triangles *AMB*, *CND*, *DPE*, *FQA* with hypotenuses *AB*, *CD*, *DE*, *FA*. Prove that *MP* = *NQ* and *MP* ⊥ *NQ* (see Figure 3).

Proof. Δ*ABM* is an isosceles right-angled triangle with a hypotenuse *AB*. Then

$$(B - M)\, i = a - M \Rightarrow bi - mi = a - m \Rightarrow m(1 - i) = a - bi \Rightarrow$$

$$\Rightarrow m = \frac{a - bi}{1 - i} = \frac{a + ai - bi - bi^2}{1 - i^2} = \frac{a + ai - bi + b}{2} = \frac{a + b + i(a + b)}{2}. \qquad (5)$$

Analogously,

$$n = \frac{c + d + i\,(c - d)}{2}, \quad p = \frac{d + e + i\,(d - e)}{2}, \quad q = \frac{a + f + i\,(f - a)}{2} \qquad (6)$$

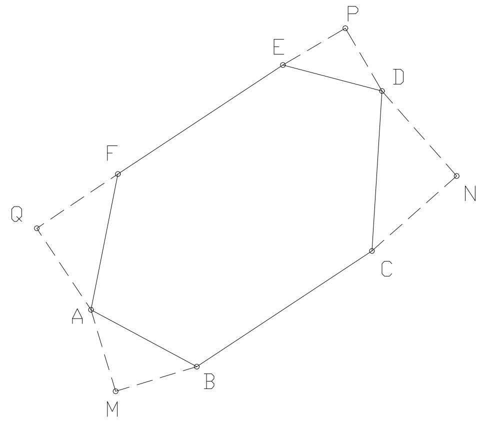


Figure 3.

We shall show that when rotating through 90° around the center of a rectangular coordinate system (the position of which does not matter) $\overrightarrow{MP}$ coincides with $\overrightarrow{NQ}$. In order to do this, we have to show that $(p-m)\,i = q-n$. From (5) and (6) we obtain consecutively:

$$i\left(\frac{d+e+i\,(d-e)}{2}-\frac{a+b+i\,(a-b)}{2}\right)=\frac{a+f+i\,(f-a)}{2}-\frac{c+d+i\,(c-d)}{2}$$

$$i((d+e)-(a+b))+(a-b)-(d-e)=(a+f)-(c+d)+i((f-a)-(c-d))$$

$$i\,(e-b)+e-b=f-c+i(f-c)$$

$$(1+i)(e-b)=(1+i)(f-c)$$

$$e-b=f-c$$

$$c-b=f-e$$

$$\overrightarrow{BC} = \overrightarrow{EF}$$

Since the last equality is valid according to the assumption of the problem, it follows that $MP = NQ$ and $MP \perp NQ$.

3. Equation of a straight line

The complex number $V\,(z_0, z_1, z_2) = \dfrac{z_0 - z_2}{z_1 - z_2}$ is called affine ratio or just ratio of the three complex numbers z_0, z_1 and z_2 (in the same order). The same number is also called ratio of the points Z_0, Z_1 and Z_2 (in the same order). The angle δ between the lines, which intersect each other at Z_0 and pass respectively through Z_1 and Z_2, equals the argument of the ratio $V\,(z_0, z_1, z_2)$.

Three points Z_0, Z_1 and Z_2 lie on a straight line if and only if the angle between the vectors $\overrightarrow{Z_1Z_2}$ and $\overrightarrow{Z_0Z_2}$ is equal to 0° or 180°. This means that the ratio $\mathbf{V}(z_0, z_1, z_2)$ is a real number, that is:

$$\frac{z_0 - z_2}{z_1 - z_2} = \frac{\overline{z_0} - \overline{z_2}}{\overline{z_1} - \overline{z_2}}$$

The straight line l, which passes through the points Z_1 and Z_2, is therefore the geometric locus of points Z, which satisfy the condition:

$$\frac{z - z_2}{z_1 - z_2} = \frac{\overline{z} - \overline{z_2}}{\overline{z_1} - \overline{z_2}}$$

or equivalently:

$$(\overline{z_1} - \overline{z_2})\, z - (z_1 - z_2)\, \overline{z} + z_1 \overline{z_2} - \overline{z_1} z_2 = 0$$

In this way every straight line has an equation $B z - \overline{B}\,\overline{z} + C = 0$, $B \neq 0$. Since $\overline{C} = z_1 \overline{z_2} - \overline{z_1} z_2 = -C$ holds true, we see that C is a purely imaginary complex number.

4. Equation of a circle

In this section we shall give a necessary and sufficient condition for four points to lie on a circle. Let points Z_0, Z_1, Z_2 and Z_3 lie on a circle (see Figure 4). Then the difference between the oriented angles $\angle\{Z_0Z_2, Z_1Z_2\}$ and $\angle\{Z_0Z_3, Z_1Z_3\}$ is equal to 0° or ±180°. The ratio $\frac{V(z_0, z_1, z_2)}{V(z_0, z_1, z_3)} = \frac{z_0 - z_2}{z_1 - z_2} : \frac{z_0 - z_3}{z_1 - z_3}$ is therefore a real number.

It is clear that if the ratio $\frac{V(z_0, z_1, z_2)}{V(z_0, z_1, z_3)}$ is a real number, then z_0, z_1, z_2 and z_3 are affixes of points lying on a circle or on a straight line. The number $W(z_0, z_1, z_2, z_3) = \frac{V(z_0, z_1, z_2)}{V(z_0, z_1, z_3)}$ is called cross ratio of the four

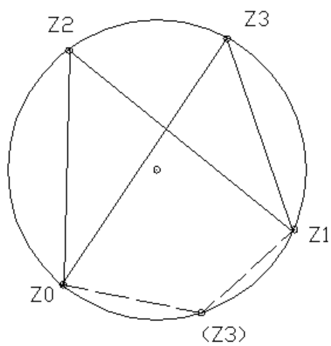


Figure 4.

affixes z_0, z_1, z_2, and z_3 (in the same order). It is therefore a necessary and sufficient condition for the four points Z_0, Z_1, Z_2 and Z_3 to lie on a circle or a straight line that the cross ratio $W(z_0, z_1, z_2, \; z_3) = \dfrac{V(z_0, z_1, z_2)}{V(z_0, z_1, z_3)}$ of the affixes z_0, z_1, z_2 and z_3 is a real number, that is:

$$\frac{z_0 - z_2}{z_1 - z_2} : \frac{z_0 - z_3}{z_1 - z_3} = \frac{\overline{z_0} - \overline{z_2}}{\overline{z_1} - \overline{z_2}} : \frac{\overline{z_0} - \overline{z_3}}{\overline{z_1} - \overline{z_3}}$$

From the last equality we can conclude that for every point Z from the circumscribed circle around $\Delta\, Z_1Z_2Z_3$ (in case that the points Z_1, Z_2, Z_3 do not lie on a straight line) the following relation holds true:

$$\frac{z - z_2}{z_1 - z_2} : \frac{z - z_3}{z_1 - z_3} = \frac{\overline{z} - \overline{z_2}}{\overline{z_1} - \overline{z_2}} : \frac{\overline{z} - \overline{z_3}}{\overline{z_1} - \overline{z_3}}$$

The latter can be considered as an equation of the circle, defined by Z_1, Z_2, Z_3. Further,

$$(z - z_2)(\overline{z} - \overline{z_3})(z_1 - z_3)(\overline{z_1} - \overline{z_2}) - (z - z_3)(\overline{z} - \overline{z}_2)(z_1 - z_2)(\overline{z_1} - \overline{z_3}) = 0$$

i .e.

$$A\, z\, \overline{z} + B\, z - \overline{B}\, \overline{z} + C = 0$$

where:

$$A = (z_1 - z_3)\,\overline{(z_1 - \overline{z_2})} - (z_1 - z_2)\,(\overline{z_1} - \overline{z_3})$$

$$B = -\overline{z_3}\,(z_1 - z_3)\,(\overline{z_1} - \overline{z_2}) + \overline{z_2}\,(z_1 - z_2)\,(\overline{z_1} - \overline{z_3})$$

$$C = z_2\,\overline{z_3}\,(z_1 - z_3)\,(\overline{z_1} - \overline{z_2}) - z_3\,\overline{z_2}\,(z_1 - z_2)\,\overline{(z_1 - \overline{z_3})}$$

Obviously, the numbers A and C are purely imaginary ($\overline{A} = -A$, $\overline{C} = -C$). In this way we obtain that every circle has an equation $A\, z\, \overline{z} + B\, z - \overline{B}\, \overline{z} + C = 0$, where A and C are purely imaginary complex numbers. It is easily seen that every equation of this type is an equation of either a circle, or a line (it is an equation of a line iff $A = 0$).

Consider the general equation of a circle:

$$A\, z\, \overline{z} + B\, z - \overline{B}\, \overline{z} + C = 0$$

where A and C are purely imaginary complex numbers. Let $A \neq 0$, i. e. we have an equation of a circle, not a line. After dividing both sides of the equation by A, we get:

$$z\,\overline{z} - a\,z - \overline{a}\,\overline{z} + b = 0$$

where $a = -\dfrac{B}{A}$ and $b = \dfrac{C}{A}$. It is obvious that b is a real number. From here we get $|z - \overline{a}|^2 = |a|^2 - b$, which shows that the point with affix $\overline{a}$ is the center of the circle and $R + \sqrt{|a|^2 - b}$ is its radius. When the center coincides with the center of the coordinate system and the radius equals 1, the equation is turned into $z\,\overline{z} = 1$. This circle is called the unit circle.

5. Unit circle

As we have seen above, the conditions for perpendicularity or parallelism of two line segments Z_1Z_2 and U_1U_2 are given by (3) and (4), that is:

$$\overrightarrow{Z_1Z_2} \perp \overrightarrow{U_1U_2} \Leftrightarrow (z_2 - z_1)\cdot(\overline{u_2} - \overline{u_1}) + (\overline{z_2} - \overline{z_1})\cdot(u_2 - u_1) = 0$$

$$\overrightarrow{Z_1Z_2} \parallel \overrightarrow{U_1U_2} \Leftrightarrow (z_2 - z_1)\cdot(\overline{u_2} - \overline{u_1}) = (\overline{z_2} - \overline{z_1})\cdot(u_2 - u_1)$$

If the line segments Z_1Z_2 and U_1U_2 are chords in the unit circle, the conjugated numbers $\overline{z_1}, \overline{z_2}, \overline{u_1}, \overline{u_2}$ in the above expressions can be replaced by $\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{u_1}, \frac{1}{u_2}$ respectively. Therefore,

$$\overrightarrow{Z_1Z_2} \perp \overrightarrow{U_1U_2} \Leftrightarrow (z_2 - z_1)\left(\frac{1}{u_2} - \frac{1}{u_1}\right) + \left(\frac{1}{z_2} - \frac{1}{z_1}\right)(u_2 - u_1) = 0 \Leftrightarrow.$$

$$\Leftrightarrow z_1 z_2 + u_1 u_2 = 0$$

Analogously,

$$\overrightarrow{Z_1Z_2} \parallel \overrightarrow{U_1U_2} \Leftrightarrow z_1 z_2 = u_1 u_2$$

The condition for three points A, B, U to lie on a straight line is:

$$\frac{u-a}{b-a} = \frac{\overline{u}-\overline{a}}{\overline{b}-\overline{a}}$$

If A and B are points from the unit circle, we have $\overline{a} = \frac{1}{a}$, $\overline{b} = \frac{1}{b}$ and then the condition is:

$$a + b = u + a\, b\, \overline{u}$$

which is a necessary and sufficient condition for U to lie on the line AB.

If Z_1Z_2 and U_1U_2 are two chords in the unit circle which intersect each other at S, then:

$$z_1 + z_2 = s + z_1\, z_2\, \overline{s}$$

$$u_1 + u_2 = s + u_1\, u_2\, \overline{s}$$

After eliminating $\overline{s}$ in the above two equalities, we get the affix s of the intersecting point S:

$$s = \frac{(z_1 + z_2)\, u_1\, u_2 - (u_1 + u_2)\, z_1\, z_2}{u_1\, u_2 - z_1\, z_2}$$

Let Z and U be two points from the unit circle, such that ZU is not a diameter. We draw the tangents to the circle at these points and let S be their intersection point. Note that S can be defined as the point, for which $SZ \perp OZ$ and $SU \perp OU$. The condition for perpendicularity imply consecutively:

$$\frac{z-s}{z} + \frac{\overline{z}-\overline{s}}{\overline{z}} = 0 \text{ and } \frac{u-s}{u} + \frac{\overline{u}-\overline{s}}{\overline{u}} = 0$$

or $(z - s)\,\overline{z} + (\overline{z} - \overline{s})\, z = 0$ and $(u - s)\,\overline{u} + (\overline{u} - \overline{s})\, u = 0$, i. e.

$$s\,\overline{z}+\overline{s}\,z=2 \text{ and } s\,\overline{u}+\overline{s}\,u=2$$

The last two equalities are called equations of the tangents *SZ* and *SU* to the circle at the points *Z* and *U* respectively.

Solving these two equations we get $s=\dfrac{2\,z\,u}{z+u}$.

We will derive a formula for the affix of the foot *S* of the perpendicular from an arbitrary point *M* to the line *AB*, where *A* and *B* are points on the unit circle. Since *S* lies on the line *AB*, it follows that:

$$a+b=s+ab\overline{s}$$

Further, MS⊥AB is equivalent to $(m-s)\,(\overline{a}-\overline{b})+(\overline{m}-\overline{s})\,(a-b)=0$, therefore $\overline{s}=\overline{m}-(\overline{m}-s)\,\overline{a}\,\overline{b}$. Replacing $\overline{s}$ in the equality $a+b=s+ab\overline{s}$ gives:

$$s=\frac{1}{2}\,(a+b+m-ab\overline{m})$$

Now we will derive a formula for the affix of the orthocenter of a triangle with vertices on the unit circle. Let $\Delta A_1A_2A_3$ be inscribed in the unit circle with center *O*. Then $|a_1|=|a_2|=|a_3|=1$. The point H_3 (see Figure 5), which is symmetric to *O* with respect to the line A_1A_2, is a vertex of the rhombus $A_1H_3A_2O$, which means $h_3 = a_1 + a_2$. We take a point *H*, so that OA_3HH_3 is a rhombus, that is $h = h_3 + a_3 = a_1 + a_2 + a_3$; moreover, $OH_3 \| A_3H$ and $OH_3 \perp A_1A_2$ hold true. Therefore, $A_3H \perp A_1A_2$, which means that *H* lies on the altitude through A_3.

Analogously, it can be shown that *H* lies on the altitudes through A_1 and A_2 as well. Therefore *H* is the orthocenter of $\Delta A_1A_2A_3$ and $h = a_1 + a_2 + a_3$.

The affix of the centroid *G* of $\Delta A_1A_2A_3$ is $g=\dfrac{a_1+a_2+a_3}{3}$. This means that *G* lies on the line *OH* and $|OG| = \dfrac{1}{3}\,|OH|$. The line *OH* is called Euler's straight line for $\Delta A_1A_2A_3$.

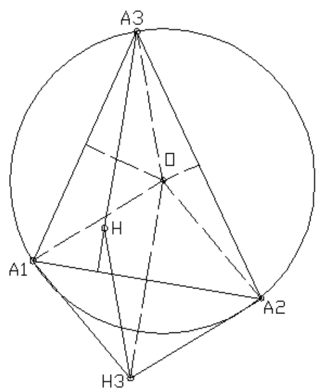


Figure 5.

Besides the center of the circumscribed circle O, the centroid G and the orthocenter H, still another remarkable point E with affix $e = \dfrac{a_1 + a_2 + a_3}{2}$ lies on the Euler's straight line. E is defined as the intersection point of the diagonals A_3H_3 and OH of the rhombus OH_3HA_3. The straight line M_3C_3 passes through it, where M_3 is the midpoint of A_1A_2 and C_3 is the midpoint of HA_3.

It is immediately seen that $|EM_3| = |EC_3| = \frac{1}{2}|OA_3| = \frac{1}{2}$, i. e. the circle k_1 with center E and radius $\frac{1}{2}$ passes through the midpoint M_3 of the line segment A_1A_2 and the midpoint C_3 of the line segment HA_3. Analogously, the circle k_1 passes through the points M_1, M_2, C_1 and C_2, where M_1 is the midpoint of A_2A_3, M_2 the midpoint of A_1A_3, C_1 the midpoint of A_1H, and C_2 the midpoint of A_2H. Since $\angle M_3P_3C_3$ is a right angle, and M_3C_3 is a diameter in the circle k_1 then $P_3 \in k_1$. Analogously, $P_1, P_2 \in k_1$ as well. The circle k_1 is called the nine points circle or the Euler-Feuerbach's circle.

The next Problem 2 is a generalization of a problem given at the XIII Russian Olympiad [1].

Problem 2. A circle k with center O and a circle intersect each other at points A and B. Let C be an arbitrary point on the arc of k, which lies inside k_1. Denote by E and D the intersection points of AC and BC with k_1, which are different from A and B, respectively. Prove that the lines DE and OC are perpendicular.

Proof. Without loss of generality we may assume that k is the unit circle (see Figure 6). As $D \in BC$, then $b + c = d + bc\overline{d}$ and therefore $\overline{d} = \dfrac{b + c - d}{bc}$. Similarly, $\overline{e} = \dfrac{a + c - e}{ac}$ is valid. Further, without any restriction we may choose $a = 1$. Then $\overline{e} = \dfrac{1 + c - e}{c}$ is satisfied. Since A, B, E and D to lie on a circle, we have $\angle ABD = \angle AED$, and:

$$\operatorname{Arg}\left(\frac{d-b}{a-b}\right) = \operatorname{Arg}\left(\frac{d-e}{a-e}\right)$$

$$\operatorname{Arg}\left(\frac{d-b}{a-b} : \frac{d-e}{a-e}\right) = 0$$

$$\operatorname{Arg}\left(\frac{(d-b)(a-e)}{(a-b)(d-e)}\right) = 0$$

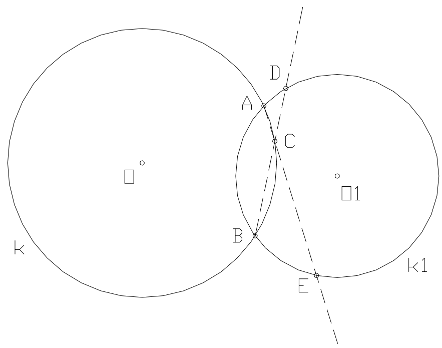


Figure 6.

The last equality means that the complex number $P=\dfrac{(d-b)(a-e)}{(a-b)(d-e)}$ is real, i. e. $P=\overline{P}$. With $a = 1$ we thus obtain:

$$P=\frac{(d-b)(1-e)}{(1-b)(d-e)} \qquad (7)$$

We have consecutively:

$$\overline{P}=\frac{(\overline{d}-\overline{b})(1-\overline{e})}{(1-\overline{b})(\overline{d}-\overline{e})}$$

$$\overline{P}=\frac{\dfrac{b+c-d}{bc}-\dfrac{1}{b}}{1-\dfrac{1}{b}}\cdot\frac{1-\dfrac{1+c-e}{c}}{\dfrac{b+c-d}{bc}-\dfrac{1+c-e}{c}}$$

$$\overline{P}=\frac{\dfrac{b+c-d-c}{bc}}{\dfrac{b-1}{b}}\cdot\frac{\dfrac{c-1-c+e}{c}}{\dfrac{b+c-d-b-bc+be}{bc}}$$

$$\overline{P}=\frac{b-d}{(b-1)\,c}\cdot\frac{(e-1)\,b}{c-d-bc+be}$$

Since $P=\overline{P}$ is valid, the latter equality and (7) lead to:

$$\frac{(d-b)(1-e)}{(1-b)(d-e)}=\frac{b-d}{(b-1)\,c}\cdot\frac{(e-1)\,b}{c-d-bc+be}$$

$$-\frac{1}{d-e}=\frac{1}{c}\cdot\frac{b}{c-d-bc+be}$$

$$c\,(c-d-bc+be)=b\,(e-d)$$

$$c^2-dc-bc^2+bec=be-bd$$

$$c^2-bc^2+bec-be=d\,(c-b)$$

$$d=\frac{c^2-bc^2+bec-be}{c-b}$$

The angle between the lines CO and DE equals the argument of the complex number $Q = \frac{d-e}{c-o} = \frac{d-e}{c}$. After substituting d in the expression for Q we obtain:

$$Q = \frac{\frac{c^2 - bc^2 + bec - be}{c-b} - e}{c} = \frac{\frac{c^2 - bc^2 + bec - be - ce + be}{c-b}}{c} = \frac{c - bc + be - e}{c-b} \tag{8}$$

Further, $\overline{Q} = \frac{\overline{c} - \overline{b}\,\overline{c} + \overline{b}\,\overline{e} - \overline{e}}{\overline{c} - \overline{b}}$, and:

$$\overline{Q} = \frac{\frac{1}{c} - \frac{1}{bc} + \frac{1+c-e}{bc} - \frac{1+c-e}{c}}{\frac{1}{c} - \frac{1}{b}} = \frac{\frac{b-1+1+c-e-b-bc+be}{bc}}{\frac{b-c}{bc}} = \frac{c - bc + be - e}{b-c} \tag{9}$$

From (8) and (9) it follows finally that $Q = -\overline{Q} \Rightarrow \angle(CO, DE) = 90° \Leftrightarrow CO \perp DE$.

The next generalization of Problem 2 is proposed by the author.

Generalized Problem 2. A circle k with center O and a circle k_1 intersect each other at points A and B. Let C be an arbitrary point on the arc from k, which lies outside k_1. Denote by E and D the intersection points of AC and BC with k_1, which are different from A and B, respectively (see Figure 7). Prove that the straight lines DE and OC are perpendicular.

6. Similar triangles

Let $A_1A_2A_3$ and $B_1B_2B_3$ be two oriented in the same way similar triangles such that the angles at the vertexes A_1, A_2 and A_3 are respectively equal to the vertexes at the edges B_1, B_2 and B_3. Then the ratios $\frac{|A_1A_3|}{|A_1A_2|}$ and $\frac{|B_1B_3|}{|B_1B_2|}$ are equal. The angle at the vertex A_1 is equal to the argument of the complex $k = \frac{a_3 - a_1}{a_2 - a_1}$. On the other hand, $k = \frac{b_3 - b_1}{b_2 - b_1}$ and it follows that $\frac{a_3 - a_1}{a_2 - a_1} = \frac{b_3 - b_1}{b_2 - b_1}$.

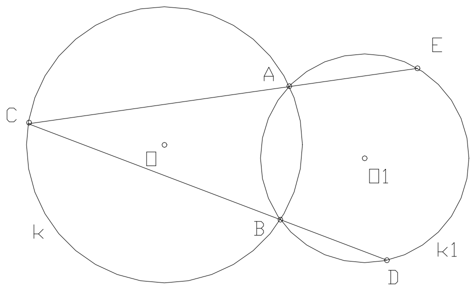


Figure 7.

If $\Delta A_1A_2A_3$ and $\Delta B_1B_2B_3$ are similar, but oriented in opposite directions, the ratios $\frac{a_3 - a_1}{a_2 - a_1}$ and $\frac{b_3 - b_1}{b_2 - b_1}$ have the same moduli and opposite arguments. Hence these ratios are the conjugated complex numbers.

7. Regular polygons

We get regular polygons when interpreting the roots of the equation $x_n = 1$ geometrically, i.e. through the numbers $w_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$, $k =$ $= 0, 1, \ldots, n$-1. Moivre's formula shows that:

$$w_0^k = (\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n})^k = w_k$$

This way of presentation is convenient when solving some problems, because the given polygon is inscribed in the unit circle. We often encounter situations where there is more than one regular polygon and then the choice of the coordinate system cannot help so much.

The following criterion (in case of any coordinate system) shows when three points A_1, A_2, A_3 are vertexes of a positively oriented equilateral triangle.

Three points A_1, A_2, A_3 are vertexes of a positively oriented equilateral triangle if and only if $A_1A_2 = A_1A_3$ and the oriented angle, that we get, when rotating A_1A_2 around A_1 to position A_1A_3, is equal to $\frac{\pi}{3}$, i. e.

$$a_3 - a_1 = \omega\,(a_2 - a_1)$$

where $\omega = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. Therefore, if a_1 and a_2 are two given complex numbers, then the number $a_3 = a_1 + \omega\,(a_2 - a_1)$ is the affix of the third vertex of the positively oriented equilateral $\Delta A_1A_2A_3$, constructed on the line segment A_1A_2.

It is important to mention that the number ω is a root of the equations $\omega^2 - \omega + 1 = 0$, $w^3 = -1$.

Problem 3. A square *ABCD* is given and four circles $\alpha(A)$, $\beta(B)$, $\gamma(C)$, $\delta(D)$ with different radii are drawn (see Figure 8). The points $X \in \alpha(A)$, $Y \in \beta(B)$, $Z \in \gamma(C)$, $T \in \delta(D)$ are moving at the same angle speed in the positive direction (counter-clockwise), such that they are vertexes of a square at a particular moment. Prove that throughout the whole movement, *X*, *Y*, *Z*, *T* are vertexes of a square.

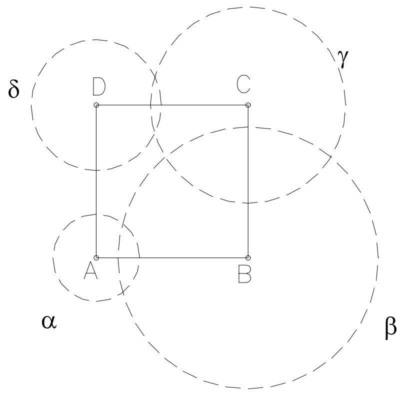


Figure 8.

The following generalization of Problem 3 is proposed and solved by the author.

Generalized Problem 3. A regular polygon $A_1 A_2 A_3 \ldots A_n$ is given. Circles $k_i(A_i)$, $1 \le i \le n$, with different radii are drawn. On every circle k_i a point B_i is moving with the same angle speed in the positive direction (counter-clockwise). Assume that in a particular moment $B_1 B_2 B_3 \ldots B_n$ is a regular polygon. Prove that throughout the movement $B_1 B_2 B_3 \ldots B_n$ remains a regular polygon.

First, we formulate a lemma which will help us to solve the generalized problem.

Lemma. Let $A_1 A_2 A_3 \ldots A_n$ and $B_1 B_2 B_3 \ldots B_n$ be congruent regular polygons, inscribed in a given circle. Then $\angle A_j O B_j$ is constant for every $j = 1, 2, \ldots n$.

Proof of the generalized problem 3. We may assume that the circumscribed circle through $A_1 A_2 A_3 \ldots A_n$ is the unit circle (see Figure 9). Therefore the vertexes have affixes which are roots of the equation $x^n = 1$ and without loss of generality we may assume that these affixes are $1, x, x^2, \ldots x^{n-1}$.

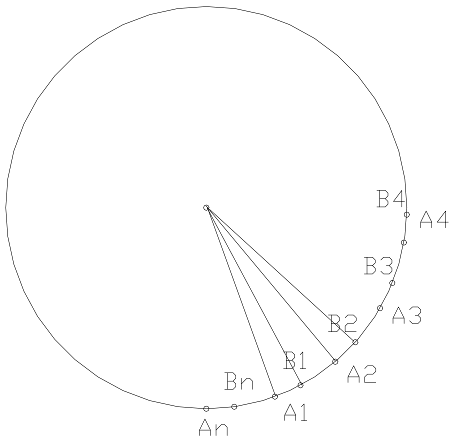


Figure 9.

Let us denote by subscripts 0 the moment when the points B_i form a regular polygon. We denote this polygon by $B_{10}\, B_{20}\, \dots B_{n\,0}$ and its center by P.

Applying a rotating homothety $h(P; k = \frac{1}{PB_{j\,0}})$ $(k \in \mathbb{R}\backslash\{0\}$, on $B_{10}\, B_{20}\, \dots B_{n\,0}$ we get a regular polygon $C_{10}C_{20}\, \dots\, C_{n\,0}$:

$$h(P; k = \frac{1}{PB_{j\,0}}) : B_{10}\, B_{20}\, \dots B_{n\,0} \rightarrow C_{10}\, C_{20}\, \dots C_{n\,0}$$

The new regular polygon $C_{10}\, C_{20}\, \dots C_{n\,0}$ can be inscribed in the unit circle because $PC_{j\,0} = \frac{1}{PB_{j\,0}} \cdot PB_{j\,0} = 1$. Then after applying a translation $T_{\overrightarrow{PQ}}$, $C_{10}\, C_{20}\, \dots C_{n\,0}$ turns into a regular polygon $D_{10}\, D_{20}\, \dots D_{n\,0}$ which is inscribed in the unit circle as well. But $A_1\, A_2\, \dots A_n$ is also a regular polygon, inscribed in the unit circle. According to the Lemma, $d_{j0} = z\, a_j$ holds true, where $z = \cos\alpha + i\sin\alpha$. Therefore,

$$d_{j0} = z\, x^{j-1} \tag{10}$$

Consider the geometric transformations by means of complex numbers:

$$h(P; k = \frac{1}{PB_{j\,0}}) : B_{j\,0} \rightarrow C_{j\,0} \Rightarrow \overrightarrow{PC_{j\,0}} = k \cdot \overrightarrow{PB_{j\,0}} \Rightarrow$$

$$\Rightarrow\ c_{j\,0} - p = k\,(b_{j\,0} - p) \Rightarrow c_{j\,0} = k\, b_{j\,0} - kp + p \tag{11}$$

$T_{\overrightarrow{PQ}} : C_{j\,0} \rightarrow D_{j\,0} \Rightarrow \overrightarrow{C_{j\,0}D_{j\,0}} = \overrightarrow{PO} \Rightarrow d_{j\,0} - c_{j\,0} = o - p$

It follows then from (11) that:

$$d_{j\,0} = k\, b_{j\,0} - k\,p + p - p \Rightarrow d_{j\,0} = k\, b_{j\,0} - k\,p \tag{12}$$

Further, (11) and (12) imply:

$$z\, x^{j-1} = k\, b_{j\,0} - k\,p \Rightarrow b_{j\,0} = \frac{z\, x^{j-1} + k\,p}{k} \tag{13}$$

Let us now assume that the points B_{j0} have been rotated through an angle φ to the points B_{j1}. Then $(a_j - b_{j0})\,\varepsilon = a_j - b_{j1}$, where $\varepsilon = \cos\varphi + i\sin\varphi$ and thus:

$$b_{j1} = a_j + (b_{j0} - a_j)\,\varepsilon.$$

From (13) it follows consecutively:

$$b_{j1} = a_j + \left(\frac{z\,a_j + k\,p}{k} - a_j\right)\varepsilon \;\Rightarrow\; b_{j1} = \frac{z\,a_j\,\varepsilon + k\,p\,\varepsilon - a_j\,\varepsilon\,k + a_j\,k}{k}$$

$$\Rightarrow b_{j1} = a_j\,\frac{z\,\varepsilon - \varepsilon\,k + k}{k} + p\,\varepsilon$$

The multiplication by $\frac{z\,\varepsilon - \varepsilon\,k + k}{k}$ presents a composition of a homothety with center O and a rotation with center O. Therefore the given polygon $A_1 A_2 \ldots A_n$ transforms into another new regular polygon M. When adding $p\varepsilon$, we translate M by the vector of translation $\overrightarrow{OV}$, where $v = p\varepsilon$. In this way M is transformed into a congruent regular polygon M_1, which in fact is $B_{11} B_{21} \ldots B_{n1}$. Therefore the polygon with vertexes B_{j1} is regular. This proves the generalized problem 3.

8. Conclusion

The paper is devoted to solving geometric problems by means of complex numbers. First, the necessary definitions and concepts of complex numbers, circles etc. are introduced. Related problems are formulated and solved. Two generalizations of known problems are formulated and solved by the author as well.

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